JOURNAL OF DIFFERENTIAL EQUATIONS 2, 33-46 (1966)

An Invariant Surface for Some Linear Singularly Perturbed Systems with Time Lag

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Received January 7, 1965

We consider a linear system with time lag of the form

$$\begin{aligned} \dot{y}(t) &= A(t)\,y(t) + B(t)\,z(t) + h(t) \\ \mu\dot{z}(t) &= C(t)\,z(t) + D(t)\,z(t-\mu\tau) + \mu F(t)\,y(t) + \mu G(t)\,y(t-\theta) + \mu H(t). \end{aligned} \tag{1}$$

We suppose that $\mu > 0$, y, h are n-vectors, z, H are m-vectors, A is an $m \times n$ matrix, C, D are $m \times n$ matrices, B is an $n \times m$ matrix, F, G are $m \times n$ matrices; functions A, B, h, C, D, F, G, H are uniformly bounded for all real t. We suppose further that the roots of the characteristic equation $\det(C + De^{-\lambda\tau} - \lambda E) = 0$ (E is the unit matrix) all lie in the half-plane $\operatorname{Re}\lambda \leqslant -2\alpha < 0$.

We shall prove that for small μ a linear function $z = L(t, \mu)\,y + g(t, \mu)$ exists such that:

1. $L(t, \mu)$, $g(t, \mu)$ are uniformly bounded for all real t.

2. If A, B, h, C, D, F, G, H are almost-periodic, then L, g are almost periodic; if A, B, h, C, D, F, G, H are periodic, then L, g are periodic.

3. $\lim_{\mu\to 0} L(t, \mu) = 0$, $\lim_{\mu\to 0} g(t, \mu) = 0$.

4. For all t_0, y_0, the functions $y(t, t_0, y_0)$, $L(t, \mu)\,y(t, t_0, y_0) + g(t, \mu)$ represent a solution of system (1) defined for all t; here $y(t, t_0, y_0)$ is the solution of the system

$$\dot{y} = [A(t) + B(t)\,L(t, \mu)]\,y + B(t)\,g(t, \mu) + h(t) \tag{2}$$

such that $y(t_0, t_0, y_0) = y_0$.

5. If $\theta = \mu\tau_0$ the invariant surface is attractive, i.e., for all solutions $y(t)$, $z(t)$ of system (1) we have

$$\lim_{t\to\infty} [z(t) - L(t, \mu)\,y - g(t, \mu)] = 0.$$

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The method will be that of Bogoliubov [*1*]; it was used for nonstationary systems by Mitropolski [*6*], for singularly perturbed systems by Zadiraka [*7*], and for differential-difference equations by Fodčuk [*2*]. Invariant surfaces for systems with time lag were also considered by Hale [*4*]; the same author studied linear singularly perturbed systems with time lag [*5*].

1. We shall construct the functions L, g by a process of successive approximations. The solution of system $\dot{y} = Ay + h$ is written as $Y_0(t, t_0)\, y_0 + \eta_0(t, t_0)$, where $Y_0(t, t_0)$ is a fundamental matrix of solutions of the homogeneous system, and

$$\eta_0(t, t_0) = \int_{t_0}^{t} Y_0(t, s)\, h(s)\, ds.$$

We define

$$L_1(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma)\, [F(\sigma)\, Y_0(\sigma, t) + G(\sigma)\, Y_0(\sigma - \theta, t)]\, d\sigma$$

$$g_1(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma)\, [F(\sigma)\, \eta_0(\sigma, t) + G(\sigma)\, \eta_0(\sigma - \theta, t) + H(\sigma)]\, d\sigma$$

where $Z(t, \sigma)$ is a matrix solution of system

$$\mu \dot{z}(t) = C(t)\, z(t) + D(t)\, z(t - \mu\tau)$$

such that $Z(t, \sigma) \equiv 0$ for $t < \sigma$, $Z(\sigma, \sigma) = E$.

From the hypothesis on the characteristic equation we deduce that there exists a $K > 0$ such that $|Z(t, \sigma)| \leqslant Ke^{-(\alpha/\mu)(t-\sigma)}$ for $t \geqslant \sigma$ (see [*3*], Lemma 2, p. 288). For sufficiently small $\mu > 0$, the convergence of the integrals in the definition of L_1, g_1 follows.

We have indeed, for $\sigma < t$, the estimation

$$|\, Y_0(\sigma, t)\, | \leqslant e^{\beta(t-\sigma)}, \qquad \beta = \sup |\, A(t)\, |.$$

Notice that we essentially use the fact that the first part of system (1) does not contain the time lags and thus the solutions are defined on the whole axis, if we replace $z(t)$ by a function defined on the whole axis.

If $\mu < \alpha/2\beta$, then

$$|\, L_1(t, \mu)\, | \leqslant \int_{-\infty}^{t} Ke^{-(\alpha/\mu)(t-\sigma)}[\,|\, F(\sigma)\, |\, e^{\beta(t-\sigma)} + |\, G(\sigma)\, |\, e^{\beta\theta} e^{\beta(t-\sigma)}]\, d\sigma$$

$$\leqslant K_1 \int_{-\infty}^{t} e^{[-(\alpha/\mu)+\beta](t-\sigma)}\, d\sigma = \frac{K_1 \mu}{\alpha - \mu\beta} \leqslant \frac{2K_1}{\alpha}\, \mu = K_2 \mu.$$

Analogously we have

$$| \eta_0(\sigma, t) | \leqslant \frac{\sup_t | h(t) |}{\beta} e^{\beta(t-\sigma)}$$

and deduce that

$$| g_1(t, \mu) | \leqslant K_3 \mu.$$

Further, if L_{n-1} and g_{n-1} are defined, we consider the system

$$\dot{y} = (A + BL_{n-1}) y + Bg_{n-1} + h.$$

The solution of this system is written

$$Y_{n-1}(t, t_0, \mu) y_0 + \eta_{n-1}(t, t_0, \mu)$$

where Y_{n-1} is a fundamental matrix of the homogeneous part, and

$$\eta_{n-1}(t, t_0, \mu) = \int_{t_0}^{t} Y_{n-1}(t, s, \mu) [B(s) g_{n-1}(s, \mu) + h(s)] \, ds.$$

We define

$$L_n(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma) [F(\sigma) Y_{n-1}(\sigma, t) + G(\sigma) Y_{n-1}(\sigma - \theta, t)] \, d\sigma$$

$$g_n(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma) [F(\sigma) \eta_{n-1}(\sigma, t) + G(\sigma) \eta_{n-1}(\sigma - \theta, t)] \, d\sigma.$$

Suppose that

$$| L_{n-1}(t, \mu) | \leqslant K_4 \mu, \qquad | g_{n-1}(t, \mu) | \leqslant K_5 \mu.$$

From

$$Y_{n-1}(\sigma, t, \mu) = E + \int_{t}^{\sigma} [A(u) + B(u) L_{n-1}(u, \mu)] \, Y_{n-1}(u, t, \mu) \, du$$

it follows that

$$| Y_{n-1}(\sigma, t, \mu) | \leqslant e^{(\beta + \gamma K_4 \mu)(t-\sigma)}, \qquad \gamma = \sup | B(t) |.$$

For $\mu < \beta/6\gamma K_4$ we have $\gamma K_4 \mu < \beta/6$ and $\beta + \gamma K_4 \mu < 7\beta/6$.
It follows that

$$| L_n(t, \mu) \leqslant K_1 \frac{\mu}{\alpha - (7\beta/\alpha) \mu} < \frac{12 K_1}{5\alpha} \mu = K_4 \mu \qquad \left(\text{if} \quad \mu < \frac{\alpha}{2\beta}\right)$$

and the estimation is proved by induction for $K_4 = 12K_1/5\alpha$ and

$$\mu < \min \left\{ \frac{\alpha}{2\beta}, \frac{5\beta\alpha}{72K_1} \right\}.$$

Analogously, we obtain

$$| \eta_{n-1} | \leqslant (\sup | h | + \gamma K_5 \mu) \int_\sigma^t e^{(7\beta/6)(t-u)}\, du \leqslant K_6 e^{(7\beta/6)(t-\sigma)}.$$

Hence,

$$| g_n(t, \mu) | \leqslant \int_{-\infty}^t K e^{-(\alpha/\mu)(t-\sigma)} [\sup | F | K_6 e^{(7\beta/6)(t-\sigma)}$$
$$+ \sup | G | e^{(7\beta/6)\theta} K_6 e^{(7\beta/6)(t-\sigma)} + \sup | H |]\, d\sigma$$
$$\leqslant \frac{K\mu}{\alpha} \sup | H | + \frac{K\mu}{\alpha - (7\mu\beta/6)} K_6 (\sup | F | + e^{(7\beta/6)\theta} \sup | G |) < K_5 \mu$$

for a convenient K_5 and μ sufficiently small, and the estimation for g_n is proved by induction. Further, from

$$Y_n(\sigma, t) - Y_{n-1}(\sigma, t) = \int_t^\sigma \{(A + BL_n) Y_n - (A + BL_{n-1}) Y_{n-1}\}\, du$$
$$= \int_t^\sigma A(Y_n - Y_{n-1})\, du + \int_t^\sigma B(L_n - L_{n-1}) Y_n du$$
$$+ \int_t^\sigma BL_{n-1}(Y_n - Y_{n-1})\, du$$

we deduce

$$| Y_n(\sigma, t) - Y_{n-1}(\sigma, t) |$$
$$\leqslant (\beta + \gamma K_4 \mu) \int_\sigma^t | Y_n(u, t) - Y_{n-1}(u, t) |\, du + \frac{\gamma \delta_n}{\beta} e^{\beta(t-\sigma)}$$

where we have denoted $\delta_n = \sup | L_n - L_{n-1} |$.

For

$$v(t) = e^{-\beta(t-\sigma)} | Y_n(\sigma, t) - Y_{n-1}(\sigma, t) |$$

we get

$$v(t) \leqslant \frac{\gamma\delta_n}{\beta} + (\beta + \gamma K_4 \mu) \int_\sigma^t e^{-\beta(t-\sigma)} | Y_n(u, t) - Y_{n-1}(u, t) |\, du$$
$$\leqslant \frac{\gamma\delta_n}{\beta} + (\beta + \gamma K_4 \mu) \int_\sigma^t e^{-\beta(u-\sigma)} | Y_n(u, t) - Y_{n-1}(u, t) |\, du$$
$$= \frac{\gamma\delta_n}{\beta} + (\beta + \gamma K_4 \mu) \int^t v(u)\, du.$$

It follows that

$$v(t) \leqslant \frac{\gamma\delta_n}{\beta} e^{(\beta+\gamma K_4\mu)(t-\sigma)}$$

and hence,

$$| Y_n(\sigma, t) - Y_{n-1}(\sigma, t) | \leqslant \frac{\gamma\delta_n}{\beta} e^{(2\beta+\gamma K_4\mu)(t-\sigma)}.$$

Since

$$L_{n+1} - L_n = \int_{-\infty}^{t} Z(t, \sigma) [F(\sigma) (Y_n(\sigma, t) - Y_{n-1}(\sigma, t)) + G(\sigma) (Y_n(\sigma - \theta, t) - Y_{n-1}(\sigma - \theta, t))] \, d\sigma$$

we obtain

$$| L_{n+1} - L_n | \leqslant \int_{-\infty}^{t} K e^{-(\alpha/\mu)(t-\sigma)} \Big[\sup | F | \frac{\gamma\delta_n}{\beta} e^{(2\beta+\gamma K_4\mu)(t-\sigma)} + \sup | G | \frac{\gamma\delta_n}{\beta} e^{(2\beta+\gamma K_4\mu)\theta} e^{(2\beta+\gamma K_4\mu)(t-\sigma)}\Big] \, d\sigma$$

$$\leqslant \frac{K_7\mu\delta_n}{\alpha - (2\beta + \gamma K_4\mu)\mu} \leqslant K_8\mu\delta_n \quad \text{if} \quad \mu < \frac{\alpha}{3\beta}, \quad \gamma K_4\mu < \beta.$$

It follows that $\delta_{n+1} \leqslant K_8\mu\delta_n$.

Analogously

$$\eta_n - \eta_{n-1} = \int_t^{\sigma} (Y_n - Y_{n-1}) (h + Bg_n) \, du + \int_t^{\sigma} Y_{n-1}B(g_n - g_{n-1}) \, du$$

and

$$| \eta_n - \eta_{n-1} | \leqslant K_6\gamma\delta_n \int_{\sigma}^{t} e^{(2\beta+\gamma K_4\mu)(u-\sigma)} \, du + \gamma\epsilon_n \int_{\sigma}^{t} e^{(\beta+\gamma K_4\mu)(u-\sigma)} \, du$$

$$\leqslant K_9 \, e^{3\beta(t-\sigma)} (\delta_n + \epsilon_n)$$

where we have let

$$\epsilon_n = \sup | g_n - g_{n-1} |.$$

From

$$g_{n+1} - g_n = \int_{-\infty}^{t} Z(t, \sigma) [F(\sigma) \eta_n(\sigma, t) + G(\sigma) \eta_n(\sigma - \theta, t) + H(\sigma)] \, d\sigma$$

$$- \int_{-\infty}^{t} Z(t, \sigma) [F(\sigma) \eta_{n-1}(\sigma, t) + G(\sigma) \eta_{n-1}(\sigma - \theta, t) + H(\sigma)] \, d\sigma$$

we get

$$|g_{n+1} - g_n| \leqslant \int_{-\infty}^{t} Ke^{-(\alpha/\mu)(t-\sigma)}[\sup |F| K_9(\delta_n + \epsilon_n) e^{3\beta(t-\sigma)} + \sup |G| e^{3\beta\theta} K_9(\delta_n + \epsilon_n)] \, d\sigma$$

$$\leqslant K_{10}\mu(\delta_n + \epsilon_n)$$

and, thus, $\epsilon_{n+1} \leqslant K_{10}\mu(\delta_n + \epsilon_n)$ and $\delta_{n+1} + \epsilon_{n+1} \leqslant K_{11}\mu(\delta_n + \epsilon_n)$. If $\mu < 1/K_{11}$, the uniform convergence of the sequences L_n and g_n follows.

2. Let

$$L(t, \mu) = \lim_{n\to\infty} L_n(t, \mu), \qquad g(t, \mu) = \lim_{n\to\infty} g_n(t, \mu).$$

The properties 1° and 3° follow from the estimations $|L(t, \mu)| \leqslant K_4\mu$, $|g(t, \mu)| \leqslant K_5\mu$. If $Y(t, t_0, \mu)$ is a fundamental matrix of system

$$\dot{y} = (A + BL)\, y$$

it follows that

$$Y(t, t_0, \mu) = \lim_{n\to\infty} Y_n(t, t_0, \mu),$$

the convergence being uniform on each finite interval. Analogously, if

$$\eta(t, t_0, \mu) = \int_{t_0}^{t} Y(t, s, \mu)\, [B(s)\, g(s, \mu) + h(s)] \, ds$$

we have

$$\eta(t, t_0, \mu) = \lim_{n\to\infty} \eta_n(t, t_0, \mu)$$

with uniform convergence on each finite interval.

It follows that

$$L(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma)\, [F(\sigma)\, Y(\sigma, t, \mu) + G(\sigma)\, Y(\sigma - \theta, t, \mu)] \, d\sigma.$$

We have indeed

$$L_n(t, \mu) - \int_{-\infty}^{t} Z(t, \sigma)\, [F(\sigma)\, Y(\sigma, t, \mu) + G(\sigma)\, Y(\sigma - \theta, t, \mu)] \, d\sigma$$

$$= \int_{-\infty}^{t} Z(t, \sigma)\, [F(\sigma)\, (Y_{n-1}(\sigma, t, \mu) - Y(\sigma, t, \mu)) + G(\sigma)\, (Y_{n-1}(\sigma - \theta, t, \mu) - Y(\sigma - \theta, t, \mu))] \, d\sigma$$

$$= \int_{-T}^{t} + \int_{-\infty}^{-T}.$$

For given $\epsilon > 0$ the second term is inferior to $\epsilon/2$ if $T > T(\epsilon)$ and then for $n > N(\epsilon)$ the first term is inferior to $\epsilon/2$ by the uniform convergence on the finite interval $[-T, t]$.

Analogously

$$g(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma)\,[F(\sigma)\,\eta(\sigma, t, \mu) + G(\sigma)\,\eta(\sigma - \theta, t, \mu) + H(\sigma)]\,d\sigma$$

Let now $y(t, t_0, y_0)$ be the solution of system (2) such that $y(t_0, t_0, y_0) = y_0$. We have

$$y(t, t_0, y_0) = Y(t, t_0, \mu)\,y_0 + \eta(t, t_0, \mu)$$

and

$$\begin{aligned}
&L(t, \mu)\,y(t, t_0, \mu) + g(t, \mu)\\
&= \int_{-\infty}^{t} Z(t, \sigma)\,[F(\sigma)\,Y(\sigma, t, \mu)\,y(t, t_0, y_0) + G(\sigma)\,Y(\sigma - \theta, t, \mu)\,y(t, t_0, y_0)]\,d\sigma\\
&\qquad + \int_{-\infty}^{t} Z(t, \sigma)\,[F(\sigma)\,\eta(\sigma, t, \mu) + G(\sigma)\,\eta(\sigma - \theta, t, \mu) + H(\sigma)]\,d\sigma\\
&= \int_{-\infty}^{t} Z(t, \sigma)\,\{F(\sigma)[Y(\sigma, t, \mu)\,y(t, t_0, \mu) + \eta(\sigma, t, \mu)]\\
&\qquad + G(\sigma)\,[Y(\sigma - \theta, t, \mu)\,y(t, t_0, \mu) + \eta(\sigma - \theta, t, \mu)] + H(\sigma)\}\,d\sigma\\
&= \int_{-\infty}^{t} Z(t, \sigma)\{F(\sigma)\,y(\sigma, t, y(t, t_0, y_0)) + G(\sigma)\,y(\sigma - \theta, t, y(t, t_0, y_0)) + H(\sigma)\}d\sigma\\
&= \int_{-\infty}^{t} Z(t, \sigma)\,\{F(\sigma)\,y(\sigma, t_0, y_0) + G(\sigma)\,y(\sigma - \theta, t_0, y_0) + H(\sigma)\}\,d\sigma.
\end{aligned}$$

It follows that the function

$$z(t, t_0, y_0) = L(t, \mu)y(t, t_0, y_0) + g(t, \mu)$$

is a solution of the system

$$\begin{aligned}
\dot{z}(t) = \frac{1}{\mu}\,C(t)\,z(t) + \frac{1}{\mu}\,D(t)\,z(t - \mu\tau)&\\
+ F(t)\,y(t, t_0, y_0) + G(t)\,y(t - \theta, t_0, y_0) + H(t).&
\end{aligned}$$

Thus $y(t, t_0, y_0)$, $z(t, t_0, y_0)$ is a solution of system (1). The property 4° is thus proved; we have obtained a family of solutions of system (1) defined on the whole axis by the initial vector y_0. For $\mu \to 0$ this family of solutions tends to the family $y(t, t_0, y_0)$, $z \equiv 0$ of the unperturbed system.

3. We shall now discuss the periodicity and almost periodicity properties of L and g. For this purpose let us remark that if C and D are periodic with period ω, then $Z(t+\omega, s+\omega) = Z(t, s)$ since in both members we have solutions of the same system and these solutions coincide for $t \leqslant s$.

From the periodicity of A and h we get

$$Y_0(t+\omega, t_0+\omega) = Y_0(t, t_0)$$

and

$$\begin{aligned}\eta_0(t+\omega, t_0+\omega) &= \int_{t_0+\omega}^{t+\omega} Y_0(t+\omega, s)\, h(s)\, ds \\ &= \int_{t_0}^{t} Y_0(t+\omega, \sigma+\omega)\, h(\sigma+\omega)\, d\sigma \\ &= \int_{t_0}^{t} Y_0(t, \sigma)\, h(\sigma)\, d\sigma = \eta_0(t, t_0).\end{aligned}$$

We have further

$$\begin{aligned}L_1(t+\omega, \mu) &= \int_{-\infty}^{t+\omega} Z(t+\omega, \sigma)\,[F(\sigma)\, Y_0(\sigma, t+\omega) \\ &\qquad\qquad + G(\sigma)\, Y_0(\sigma-\theta, t+\omega)]\, d\sigma \\ &= \int_{-\infty}^{t} Z(t+\omega, \sigma+\omega)\,[F(\sigma+\omega)\, Y_0(\sigma+\omega, t+\omega) \\ &\qquad\qquad + G(\sigma+\omega)\, Y_0(\sigma+\omega-\theta, t+\omega)]\, d\sigma \\ &= \int_{-\infty}^{t} Z(t, \sigma)\,[F(\sigma)\, Y_0(\sigma, t) + G(\sigma)\, Y_0(\sigma-\theta, t)]\, d\sigma = L_1(t, \mu)\end{aligned}$$

and analogously

$$g_1(t+\omega, \mu) = g_1(t, \mu).$$

Suppose that L_{n-1} and g_{n-1} are periodic of period ω; then, from the periodicity of A, B, h, it follows that

$$\begin{aligned}Y_{n-1}(t+\omega, t_0+\omega, \mu) &= Y_{n-1}(t, t_0, \mu), \\ \eta_{n-1}(t+\omega, t_0+\omega, \mu) &= \eta_{n-1}(t, t_0, \mu)\end{aligned}$$

and then the periodicity of L_n and g_n follows as above. The periodicity of L and g follows by the uniform convergence.

In the almost periodic case we shall use the estimations

$$| Y_0(t + r, s + r) - Y_0(t, s) | \leqslant \frac{1}{\beta} \sup | A(\sigma + r) - A(\sigma) | \, e^{2\beta(s-t)} \qquad \text{for} \quad s > t$$

$$| Z(t + r, s + r) - Z(t, s) | \leqslant \{\sup | C(t + r) - C(t) | + \sup | D(t + r) - D(t) |\} K_1 e^{-(\alpha/\mu)(t-s)} \qquad \text{for} \quad t \geqslant s.$$

We have

$$\begin{aligned}
&|L_1(t + r, \mu) - L_1(t, \mu)| \\
&\leqslant \int_{-\infty}^{t} | Z(t + r, \sigma + r) | \, [| F(\sigma + r) - F(\sigma) | \, | Y_0(\sigma + r, t + r) | \\
&\qquad + | F(\sigma) | \, | Y_0(\sigma + r, t + r) - Y_0(\sigma, t) |] \, d\sigma \\
&+ \int_{-\infty}^{t} | Z(t + r, \sigma + r) - Z(t, \sigma) | \, | F(\sigma) | \, | Y_0(\sigma, t) | \, d\sigma \\
&+ \int_{-\infty}^{t} | Z(t + r, \sigma + r) - Z(t, \sigma) | \, | G(\sigma) | \, | Y_0(\sigma - \theta, t) | \, d\sigma \\
&+ \int_{-\infty}^{t} | Z(t + r, \sigma + r) | \, [| G(\sigma + r) - G(\sigma) | \, | Y_0(\sigma - \theta + r, t + r) | \\
&\qquad + | G(\sigma) | \, | Y_0(\sigma + r - \theta, t + r) - Y_0(\sigma - \theta, t) |] \, d\sigma \\
&\leqslant \int_{-\infty}^{t} K e^{-(\alpha/\mu)(t-\sigma)} \Big\{ | F(\sigma + r) - F(\sigma) | \, e^{\beta(t-\sigma)} \\
&\qquad + | F(\sigma) | \frac{1}{\beta} \sup | A(s + r) - A(s) | \, e^{2\beta(t-\sigma)} \Big\} \, d\sigma \\
&+ \int_{-\infty}^{t} K_1 e^{-(\alpha/\mu)(t-\sigma)} [| F(\sigma) | \, e^{\beta(t-\sigma)} + | G(\sigma) | \, e^{\beta\theta} e^{\beta(t-\sigma)}] \, d\sigma \\
&\qquad \times \{\sup | C(t + r) - C(t) | + \sup | D(t + r) - D(t) |\} \\
&+ \int_{-\infty}^{t} K e^{-(\alpha/\mu)(t-\sigma)} \Big\{ | G(\sigma + r) - G(\sigma) | \, e^{\beta\theta} e^{\beta(t-\sigma)} \\
&\qquad + | G(\sigma) | \frac{1}{\beta} e^{2\beta\theta} \sup | A(s + r) - A(s) | \, e^{2\beta(t-\sigma)} \Big\} \, d\sigma \\
&\leqslant \frac{K_{12}\mu}{\alpha - 2\mu\beta} \{\sup | F(\sigma + r) - F(\sigma) | + \sup | G(\sigma + r) - G(\sigma) | \\
&+ \sup | C(\sigma + r) - C(\sigma) | \\
&+ \sup | D(\sigma + r) - D(\sigma) | + \sup | A(\sigma + r) - A(\sigma) |\}
\end{aligned}$$

and L_1 is thus almost-periodic.

In order to obtain the almost periodicity of g, we use the estimation

$$| \eta_0(t + r, t_0 + r) - \eta_0(t, t_0) | \leqslant K_{13}\{\sup | h(\sigma + r) - h(\sigma) | + \sup | A(\sigma + r) - A(\sigma) |\} e^{2\beta |t-t_0|}$$

If L_{n-1} and g_{n-1} are almost periodic the same estimations prove that L_n and g_n are almost periodic and thus, by the uniform convergence, L and g are almost periodic, and the property 2° is established.

4. Let us establish the stability property. If $y(t)$, $z(t)$ is a solution of system (1), such that y is defined for $t \geqslant t_0 - \theta$, and z is defined for

$$t \geqslant \min \{t_0 - \theta, t_0 - \mu\tau\}^1$$

$$\begin{aligned} \dot{y}(t) &= A(t) y(t) + B(t) z(t) + h(t) \\ &= [A(t) + B(t) L(t, \mu)] y(t) + B(t) g(t, \mu) \\ &\qquad + h(t) + B(t) [z(t) - L(t, \mu) y(t) - g(t, \mu)]. \end{aligned}$$

Hence

$$y(t) = Y(t, t_0, \mu) y(t_0) + \int_{t_0}^{t} Y(t, s, \mu) B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] \, ds + \eta(t, t_0, \mu).$$

From the second part of system (1) it follows that

$$\begin{aligned} z(t) = Z(t, t_0) z(t_0) &+ \int_{t_0-\mu\tau}^{t_0} Z(t, \sigma + \mu\tau) D(\sigma + \mu\tau) z(\sigma) \, d\sigma \\ &+ \int_{t_0}^{t} Z(t, \sigma) \Big\{F(\sigma) \Big(Y(\sigma, t_0, \mu) y(t_0) + \eta(\sigma, t_0, \mu) \\ &\quad + \int_{t_0}^{\sigma} Y(\sigma, s, \mu) B(s) [z(s) - Ly(s) - g] \, ds\Big) \\ &\quad + G(\sigma) (Y(\sigma - \theta, t_0, \mu) y(t_0) + \eta(\sigma - \theta, t_0, \mu) \\ &\quad + \int_{t_0}^{\sigma-\theta} Y(\sigma - \theta, s, \mu) B(s) [z(s) - Ly(s) - g] \, ds\Big) + H(\sigma)\Big\} \, d\sigma. \end{aligned}$$

[1] We may obtain this solution by specifying $y(t)$ at t_0 and $z(t)$ on $[t_0 - \gamma, t_0]$, $\gamma = \max (\mu\tau, \theta)$, and determining the values of $y(t)$ on $[t_0 - \theta, t_0]$ from the first equation in (1). The author is indebted to the referee for this remark.

We have then

$$L(t, \mu) y(t) + g(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma) \{F(\sigma)(Y(\sigma, t, \mu) y(t) + \eta(\sigma, t, \mu))$$

$$+ G(\sigma) (Y(\sigma - \theta, t, \mu) y(t) + \eta(\sigma - \theta, t, \mu)) + H(\sigma)\} d\sigma.$$

However,

$$Y(\sigma, t, \mu) y(t) = Y(\sigma, t, \mu) \Big\{ Y(t, t_0, \mu) y(t_0)$$

$$+ \int_{t_0}^{t} Y(t, s, \mu) B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

$$+ \int_{t_0}^{t} Y(t, s, \mu) [B(s) g(s, \mu) + h(s)] ds \Big\}$$

$$= Y(\sigma, t_0, \mu) y(t_0) + \int_{t_0}^{t} Y(\sigma, s, \mu) B(s)[z(s) - L(s, \mu)y(s) - g(s, \mu)]ds$$

$$+ \int_{t_0}^{t} Y(\sigma, s, \mu) [B(s) g(s, \mu) + h(s)] ds$$

and

$$Y(\sigma, t, \mu) y(t) + \eta(\sigma, t, \mu)$$

$$= Y(\sigma, t_0, \mu) y(t_0) + \int_{t_0}^{t} Y(\sigma, s, \mu) B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

$$+ \int_{t_0}^{t} Y(\sigma, s, \mu) [B(s) g(s, \mu) + h(s)] ds + \int_{t}^{\sigma} Y(\sigma, s, \mu) [B(s) g(s, \mu) + h(s)] ds$$

$$= Y(\sigma, t_0, \mu) y(t_0) + \int_{t_0}^{t} Y(\sigma, s, \mu) B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

$$+ \eta(\sigma, t_0, \mu);$$

consequently,

$$L(t, \mu) y(t) + g(t, \mu) = \int_{-\infty}^{t} Z(t, \sigma) \Big\{ F(\sigma) \Big(Y(\sigma, t_0, \mu) y(t_0)$$

$$+ \int_{t_0}^{t} Y(\sigma, s, \mu) B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds + \eta(\sigma, t_0, \mu) \Big)$$

$$+ G(\sigma) \Big(Y(\sigma - \theta, t_0, \mu) y(t_0) + \int_{t_0}^{t} Y(\sigma - \theta, s, \mu)$$

$$\times \ B(s)[z(s) - L(s, \mu) y(s) - g(s, \mu)] ds + \eta(\sigma - \theta, t_0, \mu) \Big) + H(\sigma) \Big\} d\sigma$$

$$= \int_{-\infty}^{t} Z(t, \sigma) \{F(\sigma) (Y(\sigma, t_0, \mu) y(t_0) + \eta(\sigma, t_0, \mu)) + G(\sigma) (Y(\sigma - \theta, t_0, \mu) y(t_0) + \eta(\sigma - \theta, t_0, \mu)) + H(\sigma)\} d\sigma$$

$$+ \int_{t_0}^{t} \left[\int_{-\infty}^{t} Z(t, \sigma) \{F(\sigma) Y(\sigma, s, \mu) + G(\sigma) Y(\sigma - \theta, s, \mu)\} d\sigma\right] \times B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds.$$

It follows that

$$z(t) - L(t, \mu) y(t) - g(t, \mu)$$

$$= Z(t, t_0) z(t_0) + \int_{t_0 - \mu\tau}^{t_0} Z(t, \sigma + \mu\tau) D(\sigma + \mu\tau) z(\sigma) d\sigma$$

$$- \int_{-\infty}^{t} Z(t, \sigma)\{F(\sigma) (Y(\sigma, t_0, \mu) y(t_0) + \eta(\sigma, t_0, \mu)) + G(\sigma) (Y(\sigma - \theta, t_0, \mu) y(t_0) + \eta(\sigma - \theta, t_0, \mu)) + H(\sigma)\} d\sigma$$

$$- \int_{t_0}^{t} \left[\int_{-\infty}^{s} Z(t, \sigma) F(\sigma) Y(\sigma, s, \mu) d\sigma\right] B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

$$- \int_{t_0}^{t} \left[\int_{-\infty}^{s+\theta} Z(t, \sigma) G(\sigma) Y(\sigma - \theta, s, \mu) d\sigma\right] B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

$$- \int_{t_0-\theta}^{t_0} \left[\int_{t_0}^{s+\theta} Z(t, \sigma) G(\sigma) Y(\sigma - \theta, s, \mu) d\sigma\right] \times B(s) [z(s) - L(s, \mu) y(s) - g(s, \mu)] ds$$

Therefore,

$$| z(t) - L(t, \mu) y(t) - g(t, \mu) |$$

$$\leqslant K e^{-(\alpha/\mu)(t-t_0)} | z(t_0) | + l_1 \sup_{s \in [t_0 - \mu\tau, t_0]} | z | \frac{\mu}{\alpha} e^{-(\alpha/\mu)(t-t_0)}$$

$$+ l_2 \frac{\mu}{\alpha - \mu\beta_1} e^{-(\alpha/\mu)(t-t_0)} (| y(t_0) | + m)$$

$$+ l_3 \frac{\mu^2}{(\alpha - \mu\beta_1)^2} e^{(\alpha/\mu)\theta} e^{-(\alpha/\mu)(t-t_0)} \left(\sup_{s \in [t_0-\theta, t_0]} | z | + \sup_{s \in [t_0-\theta, t_0]} | y | + \sup_{s \in [t_0-\theta, t_0]} | g | \right)$$

$$+ l_4 \frac{\mu e^{-(\alpha/\mu)t}}{\alpha - \mu\beta_1} \int_{t_0}^{t} e^{(\alpha/\mu)s} | z(s) - L(s, \mu) y(s) - g(s, \mu) | ds$$

$$+ l_5 \frac{\mu e^{(\alpha/\mu)\theta}}{\alpha - \mu\beta_1} e^{-(\alpha/\mu)t} \int_{t_0}^{t} e^{(\alpha/\mu)s} | z(s) - L(s, \mu) y(s) - g(s, \mu) | ds.$$

If $v(t)$ is defined by

$$v(t) = e^{(\alpha/\mu)t} \mid z(t) - L(t, \mu)\, y(t) - g(t, \mu) \mid,$$

then

$$v(t) \leqslant l_6\, e^{(\alpha/\mu)\, t_0} e^{(\alpha/\mu)\theta} \Big(\sup_{s \in [t_0 - \theta,\, t_0]} \mid z \mid + \sup_{s \in [t_0 - \theta,\, t_0]} \mid y \mid + l_7 \Big) + \mu l_8\, e^{(\alpha/\mu)\theta} \int_{t_0}^{t} v(s)\, ds$$

and

$$v(t) \leqslant l_6\, e^{(\alpha/\mu)\, t_0} e^{(\alpha/\mu)\theta} \Big(\sup_{s \in [t_0 - \theta,\, t_0]} \mid z \mid + \sup_{s \in [t_0 - \theta,\, t_0]} \mid y \mid + l_7 \Big) \times \exp\,[\mu l_8\, e^{(\alpha/\mu)\theta} (t - t_0)].$$

It follows that

$$\mid z(t) - L(t, \mu)\, y(t) - g(t, \mu) \mid \leqslant l_6\, e^{(\alpha/\mu)\theta} \Big(\sup_{s \in [t_0 - \theta,\, t_0]} \mid z \mid + \sup_{s \in [t_0 - \theta,\, t_0]} \mid y \mid + l_7 \Big) \times \exp \left[- \left(\frac{\alpha}{\mu} - \mu l_8\, e^{(\alpha/\mu)\theta} \right) (t - t_0) \right]$$

If $\theta = \mu \tau_1$, we finally obtain

$$\mid z(t) - L(t, \mu)\, y(t) - g(t, \mu) \mid \leqslant l_9 \Big(\sup_{s \in [t_0 - \mu\tau_1,\, t_1]} \mid z \mid + \sup_{s \in [t_0 - \mu\tau_1,\, t_0]} \mid y \mid + l_7 \Big) \times \exp \left[- \left(\frac{\alpha}{\mu} - \mu l_8\, e^{\alpha \tau_1} \right) (t - t_0) \right]$$

and for $\mu^2 < \alpha / l_8 e^{\alpha \tau_1}$ the stability property is obvious.

Remark. If the system $\dot{y} = Ay$ is uniformly asymptotically stable, then system (2) has a unique solution bounded on the whole axis; to this solution corresponds a bounded solution of system (1). In the periodic or almost periodic cases we obtain the existence of a periodic or almost-periodic solution of system (1). This result is analogous to the result of Theorem II.8 of Hale [5], but the considered system has a somewhat different form.

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